Influence of Thermal Radiation on Electrons in a High Temperature Plasma

D. BISKAMP and D. PFIRSCH

Max-Planck-Institut für Physik und Astrophysik, 8 München 23, Föhringer Ring 6

Dedicated to Prof. L. BIERMANN *on his 60th birthday*

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The velocity space diffusion coefficient of electrons in a thermal radiation field is derived "classically" and compared with the corresponding FOKKER–PLANCK coefficient arising from COULOMB collisions in a plasma. The HAMILTON–JACOBI theory is used which is especially suited to this problem.

In recent time interest in relativistic plasmas has grown. In astrophysics as well as in thermonuclear fusion research one meets situations where relativistic effects must be taken into account. These are on the one side kinematical effects as retardation and magnetic interaction, on the other the influence of free radiation, especially the scattering of electromagnetic waves (COMPTON scattering) from a thermal radiation field.

Usually relativistic kinetic equations, the simplest one being the equation of BELIAEV and BUDKER [1], do not take into account this interaction of the particles with a thermal radiation field, which is of the order e^4 as the particle-particle interaction, but is not a collective effect. We shall investigate this approximation by comparing the FOKKER–PLANCK coefficient $\langle \Delta v^2 \rangle$ of electrons in a thermal radiation field with the one of electrons in a COULOMB plasma of the same temperature. We restrict our considerations to the non-relativistic case since we are only interested in the order of magnitude which relativistic effects will not change, when the electron kinetic energy is comparable with its rest mass. Extreme relativistic plasmas with much higher energies seem to exist in nature only in cases of strong FERMI degeneracy which will not be considered here.

DREICER [2] has set up a BOLTZMANN equation for electrons in a radiation field by using the electron-photon scattering cross section. The FOKKER–PLANCK coefficients which result from this equation in the case of non-relativistic radiation temperature ($k\,T \ll m\,c^2$) can also be calculated "classically" by assuming the electrons under the influence of a statistical electromagnetic field, the modes of which are uncorrelated and their intensity given by PLANCK's law. This method has the advantage that one can easily take into account the dispersion relation of electromagnetic waves in a plasma.

1. The Diffusion Coefficient of Electrons in a Thermal Radiation Field

We calculate the diffusion coefficient for an electron at rest:

$$\langle v^2 \rangle_{\text{rad}} = \lim_{t\to\infty} \frac{1}{t}\,\overline{v^2(t)}, \tag{1}$$

$v(t)$ = velocity of the electron with $v(0)=0$. The bar indicates averaging over the statistics of the radiation field. We use the HAMILTON–JACOBI theory, which is especially suited to this problem. We start with the HAMILTON–JACOBI equation for a non-relativistic electron:

$$\frac{\partial S(\boldsymbol{x},t)}{\partial t} + \frac{1}{2\,m}\left(\frac{\partial S}{\partial \boldsymbol{x}} - \frac{e}{c}\,\boldsymbol{A}(\boldsymbol{x},t)\right)^2 = 0; \tag{2}$$
$$\frac{\partial S}{\partial \boldsymbol{x}} - \frac{e}{c}\,\boldsymbol{A} = m\,\boldsymbol{v}; \qquad \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}.$$

The solution $S(\boldsymbol{x},t)$ is developed in powers of e

$$S = S_0 + S_1 + S_2 + \dots .$$

In choosing the initial condition we must however allow for a certain indetermination of the velocity because of the field fluctuations. Since we will apply the statistics of stationary black body radiation, the initial value must be such that particle and field are initially in phase, that no time point is singled out. This is the velocity the particle has if the field is switched on adiabatically

$$\boldsymbol{v}(0) = -\frac{e}{m\,c}\,\boldsymbol{A}(\boldsymbol{x},0). \tag{3}$$

[1] S. T. BELIAEV and G. I. BUDKER, Soviet Phys.-Doklady **1**, 218 [1956].

[2] H. DREICER, Phys. Fluids **7**, 735 [1964].

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This value is consistent with the initial condition $v(0)=0$, since one has

$$\frac{e^2}{m^2 c^2}\overline{A^2} \sim v_{\rm th}^2\, \alpha\, \frac{k\,T}{m\,c^2} \ll v_{\rm th}^2; \qquad \alpha = \frac{e^2}{\hbar\,c}, \qquad v_{\rm th} = \sqrt{\frac{k\,T}{m}}, \tag{4}$$

and we are interested in times when the particle reaches thermal velocity. We obtain from (2)

$$S_0 = 0, \quad S_1 = 0; \quad \frac{\partial S_2}{\partial t} = -\frac{1}{2\,m}\frac{e^2}{c^2} A^2. \tag{5}$$

Since $\overline{\frac{\partial S}{\partial \boldsymbol{x}}\cdot \boldsymbol{A}} = 0$, $\frac{e^2}{m^2 c^2}\overline{A^2} \approx 0$, only $\frac{1}{m^2}\left(\frac{\partial S_2}{\partial \boldsymbol{x}}\right)^2$ contributes to the diffusion coefficient. We therefore find up to the second order in e

$$\left(\frac{\partial S}{\partial \boldsymbol{x}}\right)^2 = \frac{e^4}{4\,m^2 c^4}\int_0^t d\tau_1 \int_0^t d\tau_2\, \nabla A^2(\tau_1)\cdot\nabla A^2(\tau_2) = m^2\, v^2(t). \tag{6}$$

If we put

$$\boldsymbol{A}(\boldsymbol{x},t) = \sum_{\boldsymbol{k},\nu} \frac{i\,c}{\omega_{\boldsymbol{k}}^{\nu}}\,\boldsymbol{E}^{\nu} \exp i\{\boldsymbol{k}\cdot\boldsymbol{x} - \omega_{\boldsymbol{k}}^{\nu}\tau\}; \qquad \boldsymbol{E}_{\boldsymbol{k}} = \boldsymbol{E}^{*}_{-\boldsymbol{k}}, \qquad \omega_{\boldsymbol{k}} = -\omega_{-\boldsymbol{k}} \tag{7}$$

($\sum_\nu$ represents sum over forward and backward running waves), assuming periodicity in a normalizing volume V, we find

$$\left(\frac{\partial S}{\partial \boldsymbol{x}}\right)^2 = \frac{e^4}{4\,m^2 c^4}\int_0^t d\tau_1 \int_0^t d\tau_2 \sum_{\substack{\boldsymbol{k}_1,\dots,\boldsymbol{k}_2' \\ \nu_1,\dots,\nu_2'}} \boldsymbol{E}_{\boldsymbol{k}_1}^{\nu_1}\cdot \boldsymbol{E}_{\boldsymbol{k}_1}^{\nu_1'*} \boldsymbol{E}_{\boldsymbol{k}_2}^{\nu_2}\cdot \boldsymbol{E}_{\boldsymbol{k}_2'}^{\nu_2'*} \times \frac{i(\boldsymbol{k}_1-\boldsymbol{k}_1')\cdot i(\boldsymbol{k}_2-\boldsymbol{k}_2')\,c^4}{\omega_{\boldsymbol{k}_1}^{\nu_1}\,\omega_{\boldsymbol{k}_1}^{\nu_1'}\,\omega_{\boldsymbol{k}_2}^{\nu_2}\,\omega_{\boldsymbol{k}_2}^{\prime\nu_2'}} \times \exp i\{(\boldsymbol{k}_1-\boldsymbol{k}_1'+\boldsymbol{k}_2-\boldsymbol{k}_2')\cdot\boldsymbol{x} - (\omega_{\boldsymbol{k}_1}^{\nu_1}-\omega_{\boldsymbol{k}_1'}^{\nu_1'})\,\tau_1 - (\omega_{\boldsymbol{k}_2}^{\nu_2}-\omega_{\boldsymbol{k}_2'}^{\nu_2'})\,\tau_2\}. \tag{8}$$

We average (8) over space and over the phases of E, using randomness of phases. In the case of a thermal radiation field with temperature T the field correlations are given by:

$$\sum_{\nu,\nu'} \overline{(\boldsymbol{E}_{\boldsymbol{k}}^{\nu})^i\,(\boldsymbol{E}_{\boldsymbol{k}}^{\nu'})^{j*}} = \frac{1}{3}\,\frac{8\,\pi}{V}\,\frac{\hbar\,\omega}{e^{\hbar\omega/kT}-1}\,\delta^{ij}; \quad \omega = |\omega_{\boldsymbol{k}}|. \tag{9}$$

It is clear that after taking averages only the cosine part of the time exponential function survives. The asymptotic value is easily found. Since

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t d\tau_1\int_0^t d\tau_2 \cos(\omega_{k_1}-\omega_{k_2})(\tau_1-\tau_2) = 2\,\pi\,\delta(\omega_{k_1}-\omega_{k_2}), \tag{10}$$

we have

$$\langle \Delta v^2\rangle_{\rm rad} = \frac{e^4}{4\,m^4 c^4}\,4\,\pi\,\frac{V^2}{(2\,\pi)^6}\int d^3k_1\, d^3k_2 \sum_{i=1}^{3}\overline{|E_{k_1}^i|^2}\,\overline{|E_{k_2}^i|^2} \times \frac{k_1^2+k_2^2}{k_1^2\,k_2^2}\,\delta(\omega_{k_1}-\omega_{k_2}) = \frac{32}{3}\,\frac{1}{\pi}\left(\frac{e}{m\,c^2}\right)^4 \frac{(k\,T)^5}{\hbar^3}\int_0^\infty \frac{u^4\,du}{(e^u-1)^2}. \tag{11}$$

Here we have used $\omega_k^2 = k^2 c^2$, but we could easily have inserted the plasma dispersion relation

$$\omega_k^2 = \omega_p^2 + k^2 c^2.$$

The expression (11) is identical with the result of DREICER up to a factor $(2\pi)^{-1}$, which results because DREICER used an incorrect formula for the number of modes in a volume V. The integral in (11) is practically equal to one:

$$\int_0^\infty \frac{u^4}{(e^u-1)^2}\,du = 4!\,(\zeta(4)-\zeta(5)) = 1.0896,$$

$\zeta(x)$ = RIEMANN ζ-function.

2. Comparison with Fokker–Planck Coefficient on Account of Coulomb Collisions

We now compare (11) with the diffusion coefficient, likewise for an electron at rest, caused by pure COULOMB collisions (see e. g. SPITZER [3])

$$\langle \Delta v^2\rangle_{\rm coll} = \frac{8\,\pi\,e^4\,n}{m^2}\sqrt{\frac{2}{\pi}\,\frac{m}{k\,T}}\,\ln\Lambda; \qquad \Lambda = \frac{\lambda_{\rm D}}{e^2/k\,T}. \tag{12}$$

The ratio of both coefficients becomes:

$$R \equiv \frac{\langle \Delta v^2\rangle_{\rm rad}}{\langle \Delta v^2\rangle_{\rm coll}} = \frac{8}{3}\left(\frac{k\,T}{m\,c^2}\right)^4 \frac{1}{\nu}\,\frac{1}{\ln\Lambda}, \tag{13}$$

$\nu = n\left(\frac{2\,\pi\,\hbar^2}{m\,k\,T}\right)^{3/2}$ = parameter of degeneracy.

The line $R=1$ gives the boundary between the plasma states where the influence of a thermal radiation field (vaguely speaking the radiation pressure) can be neglected and those where it is dominant. It is useful do draw this line in a p-T diagram where the various states of an electron gas are given (HUND [4]) [5]. As we restrict our considerations to clas-

[3] L. SPITZER, Physics of Fully Ionized Gases, Interscience Publishers, New York 1956.

[4] F. HUND, Ergeb. Exakt. Naturw. **15**, 189 [1936].

[5] We are grateful to Prof. BIERMANN for proposing this representation.

sical systems (no degeneracy), we can assume the equation of state $p = n\,k\,T$. This approximation is consistent with the simplifying assumptions used by HUND. With T in degrees Kelvin, p in atmospheres and $\ln \Lambda \sim 10$ the ratio (13) becomes

$$R \approx 10^{-47}\, T^{13/2}/p\,. \tag{14}$$

In the $\log p - \log T$ diagram the curve $R=1$ corresponds to the straight line

$$\log T = \frac{2}{13} \log p + 7.2\,. \tag{15}$$

It is seen that there is a region around the point $p \approx 10^{16}$ atm, $T \approx 10^{9}$ °K ($n \approx 10^{29}$ cm^{-3}) where relativistic kinematics become important, but the influence of a thermal radiation field can be neglected.

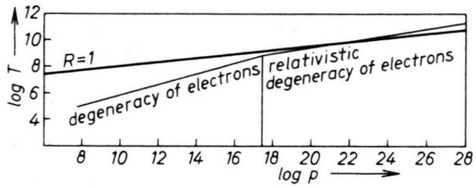

Fig. 1. Boundaries of states of a free electron gas.

From (13) one might think that one must only reduce the electron density sufficiently to get a $R>1$. But normally those dilute plasmas (belonging to the upper left part of the diagram) are not extended enough to contain the radiation emitted, so that the radiation energy density remains much smaller than that of the corresponding thermal radiation field.

Diffusion von Testteilchen in einem relativistischen Elektronenplasma

KLAUS ELSÄSSER

Max-Planck-Institut für Physik und Astrophysik, 8 München 23, Föhringer Ring 6

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The dynamical friction and diffusion coefficients of a relativistic electron plasma (kinetic energy $k \cdot T$ comparable with rest energy $m \cdot c^2$) are calculated in the LANDAU approximation. The deviations of these quantities from their classical limits are shown in Figs. 1—4 and interpreted mainly as kinematical effects due to the different relation between the kinetic energy and the velocity.

Die zeitliche Entwicklung eines klassischen Vielteilchen-Systems wird oft durch eine „kinetische Gleichung" für die Ein-Teilchen-Verteilungsfunktion in guter Näherung beschrieben. Im Falle eines Plasmas ist die LANDAU-Gleichung die einfachste (vgl. z. B. BALESCU [1]). Auch relativistische Elektronenplasmen (kinetische Energie $k\,T$ von der Größenordnung der Ruhenergie $m\,c^2$) können so beschrieben werden, solange man den Einfluß der Strahlungskorrekturen vernachlässigen kann. In einer Arbeit von BISKAMP und PFIRSCH [2] wird der Einfluß der Strahlung durch einen Vergleich der Diffusionskoeffizienten eines Elektrons in einem thermischen Strahlungsfeld bzw. in einem strahlungslosen Elektronenplasma abgeschätzt. Für große Temperaturen ist es jedoch angebracht, die relativistische Kinematik der Elektronen zu berücksichtigen. Die einfachste kinetische Gleichung, welche dies gewährleistet, ist die BELIAEV–BUDKER-Gleichung [3], welche die relativistische Form der LANDAU-Gleichung darstellt. Die Berücksichtigung der Strahlung der Elektronen würde (bei Abwesenheit äußerer Felder) nur die Formfaktoren für die Abschirmung beeinflussen, welche hier jedoch gleich eins gesetzt und durch Abschneide-Parameter des Stoßintegrals ersetzt sind. Solange also keine quantenmechanische Entartung und keine starken äußeren Felder vorliegen, ist die BELIAEV–BUDKER-Gleichung eine konsistente Beschreibung eines relativistischen Elektronenplasmas; für eine räumlich homogene Verteilungsfunktion $f(t, \boldsymbol{u})$ lautet sie:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial \boldsymbol{u}} \cdot \left\{ \boldsymbol{A}\, f + \overleftrightarrow{\boldsymbol{B}} \cdot \frac{\partial f}{\partial \boldsymbol{u}} \right\}, \tag{1}$$

[1] R. BALESCU, Statistical Mechanics of Charged Particles, Interscience Publishers, London, New York, Sydney 1963.

[2] D. BISKAMP u. D. PFIRSCH, Z. Naturforschg. **22 a**, 145 [1967].

[3] S. T. BELIAEV u. G. I. BUDKER, Soviet Phys.-Doklady **1**, 218 [1956].